The nature of self-localization in optical lattices with local dissipation

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We analyze the nature of a novel type of self-trapping transition called self-localization (SL) of Bose-Einstein condensates in one-dimensional optical lattices in the presence of weak local dissipation. SL has recently been observed in several studies based upon the discrete nonlinear Schrödinger equation (DNLS), however, its origin is hitherto an open question. We show that SL is based upon a self-trapping crossover in the system. Furthermore, we establish that the origin of the crossover is the Peierls-Nabarro barrier, an energy threshold describing the stability of self-trapped states. Beyond the mean-field description the crossover becomes even sharper which is also reflected by a sudden change of the coherence of the condensate. While the crossover can be readily studied in ongoing experiments in deep optical lattices, our results allow for the preparation of robust and long-time coherent quantum states.

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Dissipation is typically known to represent a major obstacle in the coherent control of quantum systems. However, in recent years, a strong interest in engineered dissipation has evolved, where dissipation has been used as a tool for quantum state preparation [1, 2] as well as quantum information processing and entanglement generation [3] and to induce self-trapping (ST) [4–7]. Bose-Einstein condensates (BECs) have been shown to support a variety of different kinds of ST, both in the continuous case (such as bright and dark solitons [8–13]) and in discrete systems [14–22]. A particularly high level of control has been achieved in a two-mode BEC [15], where ST can also be induced by local dissipation which can even repurify a BEC [23].

A novel self-trapping transition coined 'self-localization' has been observed numerically in several studies based upon the DNLS in the presence of weak boundary dissipation in one-dimensional deep optical lattices [24–27]. In contrast to self-trapping, where a system is either prepared in a self-trapped state [14, 15, 28] or driven towards it [4, 6, 7], SL is a mechanism where in presence of weak local or boundary dissipation a very general initially diffusive state leads to the formation of one or more discrete breathers (DBs, see [29, 30] for an overview). However, SL was only found, if the atomic interaction strength exceeds a critical value [26]. While the phenomenology of SL has been studied, the mechanisms that lead to this transition have remained unknown up to now.

In this letter, we propose a mechanism for SL allowing us to give an explicit formula for an upper bound estimate of the SL threshold for the DNLS in excellent agreement with the numerical findings of [26]. The mechanism is based on a 'crossover' which surprisingly becomes much sharper when quantum corrections beyond the mean-field description are included, which is observed, e.g., in the condensate fraction of the system.

To understand the nature of SL it is essential to note that the fixed point corresponding to the DB state into which the initial condition collapses does not undergo a bifurcation itself. On the contrary, using standard methods [31–34] the bright breather fixed point can easily be numerically found to exist and to be linearly stable for all positive nonlinearity strengths (and in the presence of boundary dissipation it becomes attractive). Linear stability analysis therefore does not suffice to understand the SL transition. The underlying idea of our approach is that near the SL threshold a single strong, localized fluctuation of the number of atoms locally brings the system's state into the vicinity of the DB fixed point in phase space. In the simplest and most likely event this strong increase in the number of atoms happens on a single site that will become the center of the DB to be formed.

We therefore study first, how a single site excitation can lead to the formation of a DB and find that there exists a distinct nonlinearity strength at which this initial condition crosses over into a self-trapped state. We show that the origin of this *ST crossover* is an energy threshold describing the stability of self-trapped states (called the Peierls-Nabarro (PN) energy barrier [22, 35]). Secondly, we statistically estimate the critical nonlinearity at the onset of SL by studying the probability that a fluctuation in a diffusive state exceeds this ST crossover and leads to the formation of a breather. The ST crossover and SL should not only be observable for BECs but as well, e.g., in coupled nonlinear optical waveguides [29, 36].

Though SL is related to ST, it is distinguished by the way in which a stable (or metastable) and spatially localized state is reached. There are several ways to obtain self-trapping of BECs in optical lattices which we classify into three types. Type I (*'static preparation'*): The quantum system is prepared in (or sufficiently close to) a self-trapped state. This has been realized in various experiments [11, 14, 15, 37]. Using a variational approach, a phase diagram has been calculated, that describes the transition from diffusion to ST for an initial Gaussian wave packet [19, 20], which, however, does not account for SL. Note that recent numerics for the DNLS [27] rather contradicts the phase diagram in [19].

Type II (*'dynamical preparation'*): Another route to ST is to apply a strong local dissipation pulse, which can depopulate one or more sites and create a stable isolated peak or vacancy [4, 6, 7, 23] (leading to the formation of a bright or dark breather). In particular, spatially resolved dissipative manipulation in an optical lattice using an electron beam with single-site addressability has been demonstrated [4]. Type III (*'self-localization'*): A third way to generate self-trapping is SL, where the system prepared in a random (generic) state in the presence of boundary or other local dissipation dynamically forms one or more DBs. Surprisingly, the locations where DBs form are not determined by the location of the leak [24–27]. In absence of boundary or local dissipation SL was not observed.

Consider the Bose-Hubbard Hamiltonian in the mean-field description [38, 39]

$$H = U\sum_{i=n}^{M}|\psi_n|^4 - \frac{J}{2}\sum_{n=1}^{M-1}(\psi_n^*\psi_{n+1} + \text{c.c.}) \qquad (1)$$

with on-site interaction U, tunneling rate J, lattice index $n = 1 \dots M$, where M denotes the number of lattice sites. Including boundary dissipation, the mean-field equations of motion are given by the dissipative DNLS (see [6, 7, 40] for a derivation of the loss term)

$$i\dot{\psi}_n = L|\psi_n|^2\psi_n - \frac{1}{2}(\psi_{n-1}+\psi_{n+1}) - i\gamma\psi_n(\delta_{n,1}+\delta_{n,M}) \quad (2)$$

with $\hbar = 1$, nonlinearity $L = (2U/J)\mathcal{N}$, dissipation rate γ, total number of atoms $\mathcal{N}$ and the normalization $\sum_n |\psi_n|^2 = 1$. We introduce a measure of the *local nonlinearity* $L_n^{\text{local}} = (2U\mathcal{N}/J)N_n$, where $N_n = |\psi_n|^2$ is the relative number of atoms (also referred to as the norm) at site n.

Let us first consider the dissipationless case with the following initial condition, where all atoms are located at site c, given by

$$\psi_n(t=0) = \delta_{nc}\,. \qquad (3)$$

In which range of the nonlinearity will the atomic population stay self-trapped at site c (resulting in the formation of a DB)? In Fig. 1 the evolution of the particle density is shown. For $L = 1.6$ (Fig. 1(a)) the particle density initially decays exponentially in time and then populates the whole lattice evenly. In contrast, a completely different behavior is observed for $L = 2.4$ in Fig. 1(b), where the initial condition relaxes into a ST state which is exponentially localized in space. A necessary condition for ST is $L_n^{\text{local}} > L_{\text{co}}$, where L_{co} is the value of the nonlinearity at the crossover that separates the diffusive from the ST regime. We define that ST is encountered, if $\min|\psi_c(t > T)|^2 > a$ for large T, which is independent of T once a breather has formed. The value for a can be estimated via the position of a saddle point (in the so-called Peierls-Nabarro energy landscape, see below) that

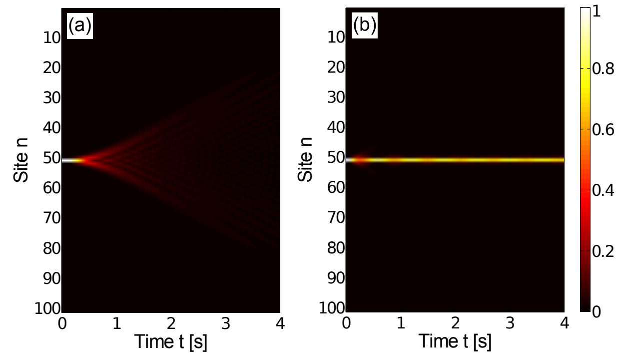

Figure 1. A ST crossover for a δ-like initial condition, where all atoms are located at a single site c, is found both beyond and within the mean-field description (depicted here for the DNLS). The color code shows the normalized atomic density $|\psi_n(t)|^2$. **(a)** Below the crossover, for $L = 1.6 < L_{\text{co}}$, the localized peak at $t = 0$ decays exponentially fast. **(b)** Above the crossover (shown is $L = 2.4 > L_{\text{co}}$) a discrete breather forms. The particle density is stable and decays exponentially in space away from the center. About 85% of the atoms are located in three sites after time $t = 4\,\text{s}$. Other parameters are $M = 101$, $c = 51$, $J = 10\,\text{Hz}$ and $\gamma = 0$.

dictates the stability of the DB [22], which is shown in Fig. 2(a). In the limit $L \to \infty$ the saddle point is found analytically at $N_2 = 1/2$ (and $N_1 = N_3 = 1/4$) [22], we therefore estimate $a = 1/2$. Starting with initial condition (3) and choosing $T = 1\,\text{s}$, the crossover from diffusion to ST is numerically found to be at $L_{\text{co}}^{\text{num}} = 2.2463$. Integration times were at least $10T$.

In the following, we will examine the ST crossover, for which we make use of a general concept called the PN energy barrier. It is given by the energy difference $|E_b - E_e|$, where E_b is the total energy of a DB centered at a single lattice site and E_e is the energy of a more extended breather centered between two lattice sites [35]. The PN barrier is based on the notion, that due to continuity, the process of translating a localized object with energy E_b from one lattice site to the adjacent one involves an intermediate state with different energy E_e.

We will connect the ST crossover to the stability of a DB. It has been shown that the stability of a DB can be well-described via a reduced problem of only few degrees of freedom [29], which reflects the fact that the breather is highly (exponentially) localized. This 'local Ansatz' has been further developed analytically in a local trimer (which is a subsystem consisting of three sites) on the so-called PN energy landscape [22], which is defined by $H_{\text{PN}} = \max_{\delta\phi_{ij}}(H)$, with $\psi_n = \sqrt{N_n}\exp(i\phi_n)$ and $\delta\phi_{ij} = \phi_i - \phi_j$ [41]. The PN landscape reads [22]

$$H_{\text{PN}} = \frac{L}{2}(N_1^2 + N_2^2 + N_3^2) + (\sqrt{N_1} + \sqrt{N_3})\sqrt{N_2}\,. \quad (4)$$

Figure 2(a) shows the PN landscape of the trimer at the ST crossover. The bright DB, which is linearly stable [42], is located in the top 'eye' of the energy landscape.

The two saddle points just below $N_2 = 1/2$ (related to a migration of the DB from site 2 to site 1 and 3 respectively) are connected to the PN barrier and the total energy threshold dictating the breather stability is given by [22]

$$E_{\rm PN}(L) = \frac{L}{4} + \frac{1}{2} + \frac{1}{4L} - \frac{1}{4L^2} + \frac{1}{4L^3} - \frac{9}{16L^4} + \mathcal{O}(\frac{1}{L^5})\,. \quad (5)$$

The energy of a bright breather E_b is a maximum of the total energy E of the trimer. As long as the total energy of the local trimer $E_{\rm PN} < E \le E_b$ is above the threshold, a breather remains pinned to a lattice site. The total energy for the initial condition (3) reads $E(L) = L/2$, which can be seen directly from Eq. (1) as the energy is measured in units of the tunneling rate J (cf. Eq. (2)). Hence, the crossover $L_{\rm co}$ is reached, when $E_{\rm PN}$ (Eq. (5)) is equal to $L/2$, and we obtain

$$L_{\rm co}^5 - 2L_{\rm co}^4 - L_{\rm co}^3 + L_{\rm co}^2 - L_{\rm co} + \frac{9}{4} = 0\,. \quad (6)$$

We find $L_{\rm co} = 2.2469$, in excellent agreement with the numerical value. This result means that the ST crossover, which is observed in a one-dimensional optical lattice, can be described with high degree of accuracy by the PN barrier of a local trimer. Given that the PN barrier describes the stability of self-trapped states in a very broad context, we expect that the three different types to obtain ST (static, dynamical and self-localized) in discrete systems eventually are related to the PN barrier.

The general behavior near the ST crossover is depicted in Fig. 2(b). The PN barrier bends off the total energy line for increasing $L > L_{\rm co}$, which leads to a growing area of stability (given by $E_0(L) > E_{\rm PN}(L)$). In the limit $L \to \infty$, the initial total energy E_0 (red line) asymptotically approaches the total energy E_b of the bright breather (blue thick line) [43]. The exact breather energy $E_b(L)$, here for $M = 101$ sites can be calculated numerically using standard methods (such as the anti-continuous limit [29, 32, 44]), while we applied a different iterative approach [34].

Beyond mean-field.– To study, how the ST crossover manifests beyond the mean-field description, we use the Bogoliubov Backreaction (BBR) method, which includes higher-order correlation functions and enables a consistent calculation of the condensate fraction of the BEC [45, 46]. The condensate fraction c_f is the fraction of the number of condensed atoms and is given by the largest eigenvalue of the reduced single-particle density matrix [38]. The BBR method has previously been generalized to the dissipative case [6, 7].

In Fig. 2(c) the minimum remaining number of atoms (normalized to 1) at the central site are shown for $M = 9$ sites and initial condition (3) using BBR and compared to the mean-field result. Boundary dissipation was applied in both cases, reducing reflections from the edges of the lattice. By including quantum corrections, the ST crossover becomes much sharper which is also reflected

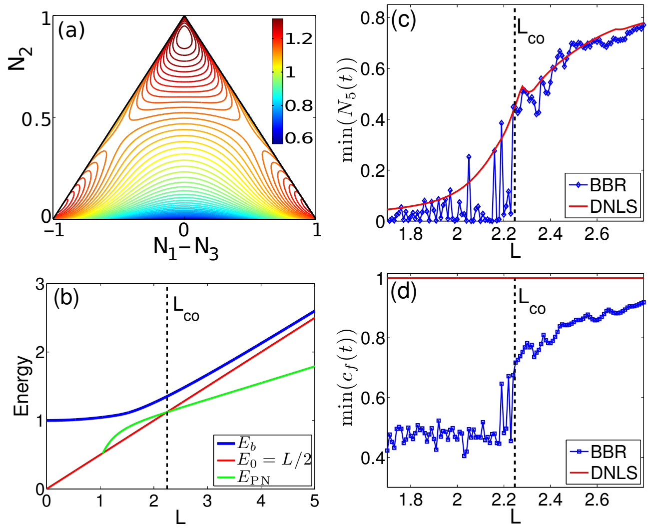

Figure 2. **(a)** The PN energy landscape $H_{\rm PN}$ exhibits a bright DB (located in the top 'eye' on the PN landscape at $N_2 = 0.902$ and $N_1 = N_3 = 0.049$) and two degenerate saddle points that mark the boundary of stability of the DB. The color code shows the PN landscape (Eq. (4)) for $L = L_{\rm co} = 2.2469$. **(b)** The ST crossover is found at the crossing (dashed line) of the total energy $E_0 = L/2$ (red line) with the PN energy barrier $E_{\rm PN}$ (Eq. (5), green line). The corresponding energy at the crossing is the total energy of the saddle points shown in (a). For $L \to \infty$, E_0 asymptotically approaches the total energy of the bright breather E_b (blue line). **(c-d)** Including quantum corrections using the BBR method (blue line), the ST crossover exhibits a much sharper transition compared to the mean-field result (red line), here shown for $M = 9$ sites. We report the minimum number of atoms at the central site min(N_5) and the condensate fraction min($c_f(t)$). The dashed line in (c-d) marks the crossover at $L_{\rm co}$ for an infinite lattice. Other parameters are $\gamma = 0.5$ Hz, $N = 200$ atoms, $J = 10$ Hz. The minima were determined in the interval $t \in [0.25, 0.5]$ sec.

by a jump in the condensate fraction, see the blue curve in Fig. 2(d). In contrast, the mean-field dynamics based upon the DNLS per se assumes a pure BEC, i.e., $c_f = 1$ (red line). While stable motion above the crossover allows for long-time coherence, unstable motion below the crossover leads to depletion of the condensate [47]. A profound understanding of the ST crossover therefore might be very useful for controlled quantum state preparation using spatially localized initial conditions, such as Eq. (3).

Self-localization.– We now turn our focus to SL, where the dynamics finds self-trapped states by itself in presence of weak boundary dissipation [24–27], resembling a phase transition [26]. To consistently investigate the dynamics in different lattices sizes M, we require the initial density $\rho = \mathcal{N}/M$ to be constant. Rescaling L accordingly results in an effective nonlinearity $\Lambda = L/M$ [26]. Starting with a homogeneous initial condition with equal norm on all lattice sites and random phases, the tran-

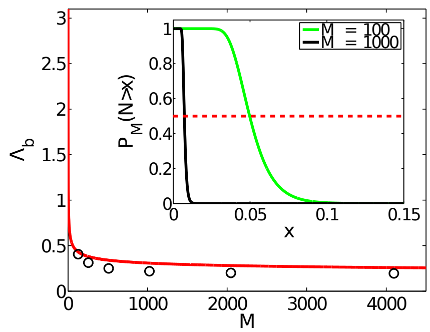

Figure 3. The SL transition at Λ_b (red line) given by Eq. (11) is an upper bound numerical results. The data (black circles) was extracted for boundary dissipation rate $\gamma = 0.2$ Hz from Fig. 1 in [26], where a sharp drop of the 'participation ratio' clearly marks the SL transition. The inset shows the probability for detecting a norm x (Eq. (9)) which exhibits a sharp transition and becomes a step function in the limit $M \to \infty$.

sition to SL has been observed at a critical interaction strength Λ_b for which we will derive an explicit expression in the following. The condition for self-trapping reads $L_n^{\mathrm{local}} = LN_n = \Lambda M N_n > L_{\mathrm{co}}$. The critical nonlinearity Λ_b for the dynamical formation of a breather is obtained for $L_n^{\mathrm{local}} = L_{\mathrm{co}}$, hence we find

$$\Lambda_b = \frac{L_{\mathrm{co}}}{M N_m} \,. \tag{7}$$

As the only unknown quantity in Eq. (7) is the maximum single site norm N_m, calculating Λ_b is reduced to a very general question: What is the probability to find a site with norm larger than a given value N in the optical lattice? In the diffusive regime, the probability distribution of norms x in the lattice is $w(x) = M\exp(-Mx)$ [26]. The probability that the norm at a certain site is smaller than x is

$$P(N < x) = \int_0^x w(x')dx' = 1 - e^{-Mx} \,. \tag{8}$$

Assuming that the populations at the M sites are independent from each other, the probability that at least one site has a norm larger than x reads

$$P_M(N > x) = 1 - [1 - e^{-Mx}]^M \,, \tag{9}$$

which approaches a step function for $M \to \infty$ (see inset of Fig. 3). Thus, the largest norm that is found in the diffusive regime is given for large M by $P_M(N > x) \approx 1/2$ (red dashed line in Fig. 3). Insertion into Eq. (9) yields

$$N_m \equiv x = \ln[\frac{1}{1-(1/2)^{1/M}}]/M \,. \tag{10}$$

With Eq. 7 the SL transition is found to be at the critical nonlinearity

$$\Lambda_b = \frac{L_{\mathrm{co}}}{\ln[\frac{1}{1-(1/2)^{1/M}}]} \,, \tag{11}$$

which is shown in Fig. 3 (red line). As in deriving Eq. (10) it was assumed that the populations at the M sites are independent, we have effectively calculated an upper bound to Λ_b, in excellent agreement with the numerical results in [26] (shown as black circles in Fig. 3).

Experiments.– We expect that the ST crossover can be readily studied in current experiments. The initial condition (3) relates to a BEC cloud at a single lattice site, while the interatomic interaction can typically be controlled with high accuracy, e.g. via a Feshbach resonance. For large lattices (such that reflections from the edges are negligible), finite size effects won't play a role (see Fig. 1). In small lattices, reflections from the edges may slightly alter L_{co} but the system still exhibits the ST crossover both in the mean-field limit and beyond. In contrast, observing SL in optical lattices is much more delicate. Weak local/boundary dissipation is a necessary prerequisite to observe SL, but could already be sufficiently present due to the inevitable coupling of the trapped condensate to the environment. The experiment, however, needs to allow for sufficient propagation time so that ST can form, in the course of which chaotic dynamics and dynamical instabilities typically lead to depletion of the condensate [37, 47, 48]. A remedy could be to reduce the relevant timescale by preparing an initial condition, that has more than exponential probability for high norms. SL represents an alternative way to induce localization where the preparation of initial wave packets is not necessary.

In conclusion, we analyzed the nature of SL in optical lattices, which previously has been observed phenomenologically in several studies [24–27], explaining recent numerical findings [26]. Our results show that the SL transition at Λ_b for which we derived an explicit estimate (Eq. (11)) is based upon two constituent parts. The first part is a ST crossover, which we studied both within and beyond the mean-field description. The second part is based on the probability, that the complex dynamics leads to a local energy above the PN energy barrier. Given the simplicity of the initial condition (3) used to probe the ST crossover, we expect that the crossover is not only experimentally readily accessible, but that its understanding could also be vital in generating long-time coherent states, without the need to fine-tune the initial state.

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